

ChebyshevT

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Notations

Traditional name

Chebyshev polynomial of the first kind

Traditional notation

 $T_n(z)$

Mathematica StandardForm notation

ChebyshevT[n , z]

Primary definition

05.04.02.0001.01

$$T_n(z) = \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n-k-1)! (2z)^{n-2k}}{k! (n-2k)!} + 2^{n-1} z^n \text{ ; } n \in \mathbb{N}$$

05.04.02.0002.01

$$T_n(z) = T_{-n}(z) \text{ ; } n \in \mathbb{Z} \wedge n < 0$$

Specific values

Specialized values

For fixed n

05.04.03.0001.01

$$T_n(0) = \cos\left(\frac{\pi n}{2}\right)$$

05.04.03.0002.01

$$T_n(1) = 1$$

05.04.03.0003.01

$$T_n(-1) = (-1)^n$$

For fixed z

05.04.03.0004.01

$$T_0(z) = 1$$

05.04.03.0005.01

$$T_1(z) = z$$

05.04.03.0006.01

$$T_2(z) = 2z^2 - 1$$

05.04.03.0007.01

$$T_3(z) = 4z^3 - 3z$$

05.04.03.0008.01

$$T_4(z) = 8z^4 - 8z^2 + 1$$

05.04.03.0009.01

$$T_5(z) = 16z^5 - 20z^3 + 5z$$

05.04.03.0010.01

$$T_6(z) = 32z^6 - 48z^4 + 18z^2 - 1$$

05.04.03.0011.01

$$T_7(z) = 64z^7 - 112z^5 + 56z^3 - 7z$$

05.04.03.0012.01

$$T_8(z) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1$$

05.04.03.0013.01

$$T_9(z) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z$$

05.04.03.0014.01

$$T_{10}(z) = 512z^{10} - 1280z^8 + 1120z^6 - 400z^4 + 50z^2 - 1$$

Values at infinities

05.04.03.0015.01

$$T_n(\infty) = \infty ; n > 0$$

05.04.03.0016.01

$$T_n(-\infty) = (-1)^n \infty ; n > 0$$

General characteristics

Domain and analyticity

The function $T_n(z)$ is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n , the function $T_n(z)$ is a polynomial in z of degree n .

05.04.04.0001.01

$$(n * z) \longrightarrow T_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.04.04.0002.01

$$T_n(-z) = (-1)^n T_n(z)$$

05.04.04.0007.01

$$T_{-n}(z) = T_n(z) ; n \in \mathbb{N}^+$$

Mirror symmetry

05.04.04.0003.01

$$T_n(\bar{z}) = \overline{T_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities**With respect to z**

The function $T_n(z)$ is polynomial and has pole of order n at $z = \infty$.

05.04.04.0004.01

$$\text{Sing}_z(T_n(z)) = \{\{\infty, n\}\}$$

Branch points**With respect to z**

The function $T_n(z)$ does not have branch points.

05.04.04.0005.01

$$\mathcal{BP}_z(T_n(z)) = \{\}$$

Branch cuts**With respect to z**

The function $T_n(z)$ does not have branch cuts.

05.04.04.0006.01

$$\mathcal{BC}_z(T_n(z)) = \{\}$$

Series representations**Generalized power series**

Expansions at generic point $z = z_0$

For the function itself

05.04.06.0018.01

$$T_n(z) \propto T_n(z_0) + n U_{n-1}(z_0) (z - z_0) + \frac{n}{2(z_0^2 - 1)} (n T_n(z_0) - z_0 U_{n-1}(z_0)) (z - z_0)^2 + \dots /; (z \rightarrow z_0)$$

05.04.06.0019.01

$$T_n(z) \propto T_n(z_0) + n U_{n-1}(z_0) (z - z_0) + \frac{n}{2(z_0^2 - 1)} (n T_n(z_0) - z_0 U_{n-1}(z_0)) (z - z_0)^2 + O((z - z_0)^3)$$

05.04.06.0020.01

$$T_n(z) = T_n(z_0) + n \sum_{k=1}^n \frac{2^{k-1}}{k} C_{n-k}^k(z_0) (z - z_0)^k$$

05.04.06.0021.01

$$T_n(z) = \sqrt{\pi} \sum_{k=0}^n \frac{(z_0 - 1)^{-k}}{k!} {}_3\tilde{F}_2\left(1, -n, n; \frac{1}{2}, 1 - k; \frac{1 - z_0}{2}\right) (z - z_0)^k$$

05.04.06.0022.01

$$T_n(z) = T_n(z_0) + n \sum_{k=1}^n \frac{2^{k-1}}{k} \left(\sum_{i_1=0}^{n-k} \dots \sum_{i_k=0}^{n-k} \delta_{\sum_{j=1}^k i_j, n=k} \prod_{j=1}^k U_{i_j}(z_0) \right) (z - z_0)^k$$

05.04.06.0023.01

$$T_n(z) \propto T_n(z_0) (1 + O(z - z_0))$$

Expansions at $z = 0$

For the function itself

05.04.06.0001.02

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + n \sin\left(\frac{\pi n}{2}\right) z - \frac{n^2}{2} \cos\left(\frac{\pi n}{2}\right) z^2 + \dots /; (z \rightarrow 0)$$

05.04.06.0024.01

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + n \sin\left(\frac{\pi n}{2}\right) z - \frac{n^2}{2} \cos\left(\frac{\pi n}{2}\right) z^2 + O(z^3)$$

05.04.06.0002.01

$$T_n(z) = F_{1 \times 0 \times 0}^{2 \times 0 \times 0} \left(\begin{matrix} -n, n, \dots; 1 \\ \frac{1}{2}, \dots; \frac{z}{2}, -\frac{z}{2} \end{matrix} \right)$$

05.04.06.0025.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{n}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} z^{2k} + n \sin\left(\frac{\pi n}{2}\right) z \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(\frac{n+1}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} z^{2k}$$

05.04.06.0026.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) {}_2F_1\left(-\frac{n}{2}, \frac{n}{2}; \frac{1}{2}; z^2\right) + n \sin\left(\frac{\pi n}{2}\right) z {}_2F_1\left(\frac{1-n}{2}, \frac{n+1}{2}; \frac{3}{2}; z^2\right)$$

05.04.06.0027.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) \cos(n \sin^{-1}(z)) + \sin\left(\frac{\pi n}{2}\right) \sin(n \sin^{-1}(z))$$

05.04.06.0028.01

$$T_n(z) = \cos\left(\frac{1}{2} n (\pi - 2 \sin^{-1}(z))\right)$$

05.04.06.0003.02

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right) (1 + O(z))$$

05.04.06.0004.01

$$T_n(z) = \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n-k-1)! (2z)^{n-2k}}{k! (n-2k)!} + 2^{n-1} z^n$$

05.04.06.0005.02

$$T_n(z) \propto (-1)^{\lfloor \frac{n}{2} \rfloor} (nz)^{n-2 \lfloor \frac{n}{2} \rfloor} (1 + O(z^2)) / ; n > 0$$

Expansions at $z = 1$ **For the function itself**

05.04.06.0006.02

$$T_n(z) \propto 1 + n^2 (z-1) - \frac{(1-n)n^2(1+n)}{16} (z-1)^2 + \dots / ; (z \rightarrow 1)$$

05.04.06.0029.01

$$T_n(z) \propto 1 + n^2 (z-1) - \frac{(1-n)n^2(1+n)}{16} (z-1)^2 + O((z-1)^3)$$

05.04.06.0007.01

$$T_n(z) = \sum_{k=0}^n \frac{(-n)_k (n)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k$$

05.04.06.0008.01

$$T_n(z) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-z}{2}\right)$$

05.04.06.0030.01

$$T_n(z) = \cos\left(2n \sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

05.04.06.0009.02

$$T_n(z) \propto 1 + O(z-1)$$

Expansions at $z = -1$ **For the function itself**

05.04.06.0010.02

$$T_n(z) \propto (-1)^n \left(1 - n^2(z+1) - \frac{(1-n)n^2(1+n)}{6} (z+1)^2 - \dots\right) / ; (z \rightarrow -1)$$

05.04.06.0031.01

$$T_n(z) \propto (-1)^n \left(1 - n^2(z+1) - \frac{(1-n)n^2(1+n)}{6} (z+1)^2\right) + O((z+1)^3)$$

05.04.06.0011.01

$$T_n(z) = (-1)^n \sum_{k=0}^n \frac{(-n)_k (n)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{z+1}{2}\right)^k$$

05.04.06.0012.01

$$T_n(z) = (-1)^n {}_2F_1\left(-n, n; \frac{1}{2}; \frac{z+1}{2}\right)$$

05.04.06.0032.01

$$T_n(z) = (-1)^n \cos\left(2n \sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

05.04.06.0013.02

$$T_n(z) \propto (-1)^n (1 + O(z+1))$$

Expansions at $z = \infty$ **For the function itself****Expansions in $1/z$**

05.04.06.0033.01

$$T_n(z) \propto 2^{n-1} z^n \left(1 - \frac{n}{4z^2} + \frac{n(n-3)}{32z^4} + \dots\right); (|z| \rightarrow \infty) \wedge n > 0$$

05.04.06.0034.01

$$T_n(z) \propto 2^{n-1} z^n \left(1 - \frac{n}{4z^2} + \frac{n(n-3)}{32z^4} + O\left(\frac{1}{z^6}\right)\right); n > 0$$

05.04.06.0035.01

$$T_n(z) = 2^{n-1} z^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k z^{-2k}}{k! (1-n)_k}; n > 0$$

05.04.06.0036.01

$$T_n(z) = 2^{n-1} z^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n; \frac{1}{z^2}\right); n > 2$$

05.04.06.0037.01

$$T_n(z) = \frac{1}{2} (-z)^{-\left(\frac{n+1}{2}\right)} z^{-\frac{n}{2}} \left(z^2 + \sqrt{z^2} \sqrt{z^2 - 1}\right)^{-n} \\ \left(\cos\left(\frac{n\pi}{2}\right) \sqrt{-z} \left(z^{2n} + (-1)^n \left(z^2 + \sqrt{z^2} \sqrt{z^2 - 1}\right)^{2n}\right) + \sin\left(\frac{n\pi}{2}\right) \sqrt{z} \left((-1)^n \left(z^2 + \sqrt{z^2} \sqrt{z^2 - 1}\right)^{2n} - z^{2n}\right)\right); n \in \mathbb{N}$$

Expansions in $1/(1-z)$

05.04.06.0014.02

$$T_n(z) \propto 2^n n (z-1)^n \left(\frac{1}{2n} + \frac{2}{z-1} + \frac{2(2n-3)}{(z-1)^2} + \dots\right); (|z| \rightarrow \infty) \wedge n > 0$$

05.04.06.0038.01

$$T_n(z) \propto 2^n n (z-1)^n \left(\frac{1}{2n} + \frac{2}{z-1} + \frac{2(2n-3)}{(z-1)^2} + O\left(\frac{1}{z^3}\right)\right); n > 0$$

05.04.06.0015.01

$$T_n(z) = 2^{n-1} (z-1)^n \sum_{k=0}^{2n-1} \frac{(-n)_k \left(\frac{1}{2} - n\right)_k}{k! (1-2n)_k} \left(\frac{2}{1-z}\right)^k; \quad n > 0$$

05.04.06.0039.01

$$T_n(z) = \frac{2^{-n} \sqrt{\pi} (z-1)^n}{(n-1)!} \sum_{k=0}^{2n-1} \frac{(2n-k-1)! (-n)_k}{k! \Gamma\left(-k+n+\frac{1}{2}\right)} \left(\frac{2}{1-z}\right)^k; \quad n > 0$$

05.04.06.0040.01

$$T_n(z) = 2^{n-1} (z-1)^n {}_2F_1\left(-n, \frac{1}{2} - n; 1-2n; \frac{2}{1-z}\right); \quad n > 1$$

05.04.06.0041.01

$$T_n(z) = 2^{n-1} (z-1)^{-n} \left(\sqrt{\frac{z+1}{z-1}} + 1 \right)^{-2n} + 2^{-n-1} (z-1)^n \left(\sqrt{\frac{z+1}{z-1}} + 1 \right)^{2n}$$

05.04.06.0016.02

$$T_n(z) \propto 2^{n-1} z^n \left(1 + O\left(\frac{1}{z}\right) \right); \quad n > 0$$

Expansions at $n = \infty$

05.04.06.0042.01

$$T_n(z) = \cos(n \cos^{-1}(z))$$

05.04.06.0043.01

$$T_n(z) \propto \begin{cases} \frac{1}{2} e^{i n \cos^{-1}(z)} & -\pi < \arg(\cos^{-1}(z)) < 0 \\ \frac{1}{2} e^{-i n \cos^{-1}(z)} & 0 < \arg(\cos^{-1}(z)) < \pi \\ \cos(n \cos^{-1}(z)) & \text{True} \end{cases}; \quad (n \rightarrow \infty)$$

Other series representations

05.04.06.0017.01

$$T_n(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} z^{n-2k} (z^2 - 1)^k$$

Integral representations

Integral representations of negative integer order

Rodrigues-type formula.

05.04.07.0001.01

$$T_n(z) = \frac{(-1)^n \sqrt{\pi} \sqrt{1-z^2}}{2^n \Gamma\left(n + \frac{1}{2}\right)} \frac{\partial^n (1-z^2)^{n-\frac{1}{2}}}{\partial z^n}$$

Limit representations

05.04.09.0001.01

$$T_n(z) = \frac{n}{2} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} C_n^\lambda(z)$$

Generating functions

05.04.11.0001.01

$$T_n(z) = \left([t^n] \frac{1 - tz}{t^2 - 2zt + 1} \right) /; -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.04.13.0001.01

$$(1 - z^2) w''(z) - z w'(z) + n^2 w(z) = 0 /; w(z) = c_1 T_n(z) + c_2 \sin(n \cos^{-1}(z))$$

05.04.13.0002.01

$$W_z(T_n(z), \sin(n \cos^{-1}(z))) = -\frac{n}{\sqrt{1 - z^2}}$$

05.04.13.0003.01

$$(1 - z^2) w''(z) - z w'(z) + n^2 w(z) = 0 /; w(z) = c_1 T_n(z) + c_2 \sinh(n \cosh^{-1}(z))$$

05.04.13.0004.01

$$W_z(T_n(z), \sinh(n \cosh^{-1}(z))) = \frac{n \cos(n \cos^{-1}(z)) \cosh(n \cosh^{-1}(z))}{\sqrt{z - 1} \sqrt{z + 1}} - \frac{n \sin(n \cos^{-1}(z)) \sinh(n \cosh^{-1}(z))}{\sqrt{1 - z^2}}$$

05.04.13.0005.01

$$(1 - z^2) w''(z) - z w'(z) + n^2 w(z) = 0 /; w(z) = c_1 T_n(z) + c_2 \sqrt{1 - z^2} U_{n-1}(z)$$

05.04.13.0006.01

$$W_z(T_n(z), \sqrt{1 - z^2} U_{n-1}(z)) = -\frac{n}{\sqrt{1 - z^2}}$$

05.04.13.0007.01

$$w''(z) - \left(\frac{g(z) g'(z)}{1 - g(z)^2} + \frac{g''(z)}{g'(z)} \right) w'(z) + \frac{n^2 g'(z)^2}{1 - g(z)^2} w(z) = 0 /; w(z) = c_1 T_n(g(z)) + c_2 \sqrt{1 - g(z)^2} U_{n-1}(g(z))$$

05.04.13.0008.01

$$W_z(T_n(g(z)), \sqrt{1 - g(z)^2} U_{n-1}(g(z))) = -\frac{n g'(z)}{\sqrt{1 - g(z)^2}}$$

05.04.13.0009.01

$$w''(z) - \left(\frac{g(z)g'(z)}{1-g(z)^2} + \frac{2h'(z)}{h(z)} + \frac{g''(z)}{g'(z)} \right) w'(z) + \left(\frac{n^2 g'(z)^2}{1-g(z)^2} + \frac{g(z)h'(z)g'(z)}{(1-g(z)^2)h(z)} + \frac{h(z)h'(z)g''(z) + g'(z)(2h'(z)^2 - h(z)h''(z))}{h(z)^2 g'(z)} \right) w(z) = 0 ;$$

$$w(z) = c_1 h(z) T_n(g(z)) + c_2 h(z) \sqrt{1-g(z)^2} U_{n-1}(g(z))$$

05.04.13.0010.01

$$W_z \left(h(z) T_n(g(z)), h(z) \sqrt{1-g(z)^2} U_{n-1}(g(z)) \right) = - \frac{n h(z)^2 g'(z)}{\sqrt{1-g(z)^2}}$$

05.04.13.0011.01

$$z^2 (a^2 z^{2r} - 1) w''(z) + (r - (2s-1)(a^2 z^{2r} - 1)) z w'(z) + (a^2 z^{2r} (s^2 - r^2 n^2) - s(r+s)) w(z) = 0 ;$$

$$w(z) = c_1 z^s T_n(a z^r) + c_2 z^s \sqrt{1-a^2 z^{2r}} U_{n-1}(a z^r)$$

05.04.13.0012.01

$$W_z \left(z^s T_n(a z^r), z^s \sqrt{1-a^2 z^{2r}} U_{n-1}(a z^r) \right) = - \frac{a r n z^{r+2s-1}}{\sqrt{1-a^2 z^{2r}}}$$

05.04.13.0013.01

$$(a^2 r^{2z} - 1) w''(z) + (\log(r) - 2(a^2 r^{2z} - 1) \log(s)) w'(z) + (a^2 r^{2z} (\log^2(s) - n^2 \log^2(r)) - \log(s)(\log(r) + \log(s))) w(z) = 0 ;$$

$$w(z) = c_1 s^z T_n(a r^z) + c_2 s^z \sqrt{1-a^2 r^{2z}} U_{n-1}(a r^z)$$

05.04.13.0014.01

$$W_z \left(s^z T_n(a r^z), s^z \sqrt{1-a^2 r^{2z}} U_{n-1}(a r^z) \right) = - \frac{a r^z n s^{2z} \log(r)}{\sqrt{1-a^2 r^{2z}}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.04.16.0001.01

$$T_n(-z) = (-1)^n T_n(z)$$

Multiple arguments

05.04.16.0005.01

$$T_{2n} \left(\sqrt{\frac{z+1}{2}} \right) = T_n(z)$$

Brychkov Yu.A. (2007)

05.04.16.0006.01

$$T_{2n}(z) = (-1)^n T_n(1-2z^2)$$

Products, sums, and powers of the direct function

Products of the direct function

05.04.16.0002.01

$$T_n(z) T_m(z) = \frac{1}{2} (T_{m+n}(z) + T_{|n-m|}(z)) ; m > 0 \wedge n > 0$$

Addition formulas

05.04.16.0004.02

$$T_n(\cos(\theta_0)) = \sum_{k=0}^n \sum_{j_1=0}^{\lfloor \frac{n-k}{2} \rfloor} \sum_{j_2=0}^{\lfloor \frac{n-k}{2} \rfloor} \sum_{j_3=0}^{\lfloor \frac{k}{2} \rfloor} \frac{n 4^k (n-k)! \left(\frac{1}{2} - k\right)}{\pi \Gamma(k+n)}$$

$$\left\{ \begin{array}{ll} 1 & k = j_1 = j_2 = 0 \\ 0 & k + j_1 = 0 \vee k + j_2 = 0 \\ \frac{\Gamma(k+j_1) \Gamma(k+j_2)}{\Gamma(k) \Gamma(k)} & \text{True} \end{array} \right. \frac{\Gamma(n-j_1)}{j_1! \Gamma(-k+n-j_1+1)} \frac{\Gamma(n-j_2)}{j_2! \Gamma(-k+n-j_2+1)} \frac{\Gamma(j_3 - \frac{1}{2}) \Gamma(k-j_3 - \frac{1}{2})}{j_3! \Gamma(k-j_3+1)}$$

$$\left(1 - \frac{1}{2} \delta_{2j_1, n-k}\right) \left(1 - \frac{1}{2} \delta_{2j_2, n-k}\right) \left(1 - \frac{1}{2} \delta_{2j_3, k}\right)$$

$$\sin^k(\theta) \sin^k(\vartheta) T_{n-k-2j_1}(\cos(\theta)) T_{n-k-2j_2}(\cos(\vartheta)) T_{k-2j_3}(\cos(\phi)) ; n \in \mathbb{N} \wedge \cos(\theta_0) = \cos(\theta) \cos(\vartheta) + \cos(\phi) \sin(\theta) \sin(\vartheta)$$

Related transformations

05.04.16.0003.01

$$T_n(T_m(z)) = T_{nm}(z) ; m > 0 \wedge n > 0$$

Identities

Recurrence identities

Consecutive neighbors

05.04.17.0001.01

$$T_n(z) = 2z T_{n+1}(z) - T_{n+2}(z)$$

05.04.17.0002.01

$$T_n(z) = 2z T_{n-1}(z) - T_{n-2}(z)$$

Distant neighbors

05.04.17.0008.01

$$T_n(z) = C_m(n, z) T_{n+m}(z) - C_{m-1}(n, z) T_{n+m+1}(z) ; C_0(n, z) = 1 \wedge C_1(n, z) = 2z \wedge C_m(n, z) = 2z C_{m-1}(n, z) - C_{m-2}(n, z) \wedge m > 0$$

05.04.17.0009.01

$$T_n(z) = C_m(n, z) T_{n-m}(z) - C_{m-1}(n, z) T_{n-m-1}(z) ; C_0(n, z) = 1 \wedge C_1(n, z) = 2z \wedge C_m(n, z) = 2z C_{m-1}(n, z) - C_{m-2}(n, z) \wedge m > 0$$

05.04.17.0003.01

$$T_n(z) = 2(-1)^2 \left\lfloor \frac{m}{2} \right\rfloor z^{m-2} \left\lfloor \frac{m}{2} \right\rfloor (z^2)^{\frac{1-m}{2} + \left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}(2z^2-1) T_{n+m}(z) - 2z^{1-m+2} \left\lfloor \frac{m}{2} \right\rfloor (z^2)^{\left\lfloor \frac{m+1}{2} \right\rfloor - \frac{m}{2}} U_{\frac{m-1}{2}}(2z^2-1) T_{n+m+1}(z) ; m > 0$$

05.04.17.0004.01

$$T_n(z) = 2(-1)^2 \left\lfloor \frac{m}{2} \right\rfloor z^{m-2} \left\lfloor \frac{m}{2} \right\rfloor (z^2)^{\frac{1-m}{2} + \left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}(2z^2-1) T_{n-m}(z) - 2z^{1-m+2} \left\lfloor \frac{m}{2} \right\rfloor (z^2)^{\left\lfloor \frac{m+1}{2} \right\rfloor - \frac{m}{2}} U_{\frac{m}{2}-1}(2z^2-1) T_{n-m-1}(z) ; m > 0$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.04.17.0005.01

$$T_{n-1}(z) + T_{n+1}(z) = 2z T_n(z)$$

05.04.17.0006.01

$$T_n(z) = \frac{1}{2z} (T_{n-1}(z) + T_{n+1}(z))$$

Normalized recurrence relation

05.04.17.0007.01

$$z p(n, z) = \frac{1}{4} p(n-1, z) + p(n+1, z) ; p(n, z) = 2^{-n} T_n(z)$$

Complex characteristics

Real part

05.04.19.0001.01

$$\operatorname{Re}(T_n(x + iy)) = T_n(x) + n \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j 2^{2j-2} y^{2j}}{j} C_{n-2j}^{2j}(x) ; x \in \mathbb{R} \wedge y \in \mathbb{R}$$

Imaginary part

05.04.19.0002.01

$$\operatorname{Im}(T_n(x + iy)) = n \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j 2^{2j} y^{2j+1}}{2j+1} C_{n-2j-1}^{2j+1}(x) ; x \in \mathbb{R} \wedge y \in \mathbb{R}$$

Argument

05.04.19.0003.01

$$\arg(T_n(x + iy)) = \tan^{-1} \left(T_n(x) + n \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j 2^{2j-2}}{j} C_{n-2j}^{2j}(x) y^{2j}, n \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j 2^{2j}}{2j+1} C_{n-2j-1}^{2j+1}(x) y^{2j+1} \right) ; x \in \mathbb{R} \wedge y \in \mathbb{R}$$

Conjugate value

05.04.19.0004.01

$$\overline{T_n(x + i y)} = T_n(x) - i n \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^j 2^{2j}}{2j+1} C_{-2j+n-1}^{2j+1}(x) y^{2j+1} + n \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j 2^{2j-2}}{j} C_{n-2j}^{2j}(x) y^{2j} ; x \in \mathbb{R} \wedge y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to z

Forward shift operator:

05.04.20.0001.01

$$\frac{\partial T_n(z)}{\partial z} = n U_{n-1}(z)$$

05.04.20.0002.01

$$\frac{\partial^2 T_n(z)}{\partial z^2} = \frac{n}{z^2 - 1} (n T_n(z) - z U_{n-1}(z))$$

Symbolic differentiation

With respect to z

05.04.20.0003.01

$$\frac{\partial^m T_n(z)}{\partial z^m} = n 2^{m-1} (m-1)! C_{n-m}^m(z) ; m \in \mathbb{N}^+$$

05.04.20.0004.02

$$\frac{\partial^m T_n(z)}{\partial z^m} = \sqrt{\pi} (z-1)^{-m} {}_3\tilde{F}_2\left(1, -n, n; \frac{1}{2}, 1-m; \frac{1-z}{2}\right) ; m \in \mathbb{N}$$

05.04.20.0006.01

$$\frac{\partial^m T_n(z)}{\partial z^m} = 2^{m-1} (m-1)! n \sum_{i_1=0}^{n-m} \dots \sum_{i_m=0}^{n-m} \delta_{\sum_{j=1}^m i_j, n-m} \prod_{j=1}^m U_{i_j}(z) ; m \in \mathbb{N}^+ \wedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

05.04.20.0005.01

$$\frac{\partial^\alpha T_n(z)}{\partial z^\alpha} = z^{-\alpha} \sqrt{\pi} \tilde{F}_{1 \times 1 \times 0}^{2 \times 1 \times 0} \left(-n, n; 1; ; -\frac{z}{2}, \frac{1}{2} \right) \left(\frac{1}{2}; 1-\alpha; ; -\frac{z}{2}, \frac{1}{2} \right)$$

Integration

Indefinite integration

Involving only one direct function

05.04.21.0001.01

$$\int T_n(a z) dz = \frac{1}{2a} \left(\frac{T_{n-1}(a z)}{1-n} + \frac{T_{n+1}(a z)}{n+1} \right) ; n \neq 1$$

05.04.21.0002.01

$$\int T_n(z) dz = \frac{1}{2} \left(\frac{T_{n-1}(z)}{1-n} + \frac{T_{n+1}(z)}{n+1} \right) ; n \neq 1$$

05.04.21.0003.01

$$\int T_n(z) dz = \frac{2^{n-1} z^{n+1}}{n+1} + \frac{1}{2} \delta_{n,0} z + \frac{1}{4} n \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n-k-1)! (2z)^{n-2k+1}}{k! (n-2k+1)!}$$

Involving one direct function and elementary functions

Involving power function

05.04.21.0004.01

$$\int z^{\alpha-1} T_n(z) dz = \frac{\delta_{n,0} z^{\alpha+1}}{2(\alpha+1)} + \frac{n}{2} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n-k-1)! 2^{n-2k} z^{n-2k+\alpha}}{k! (n-2k)! (n-2k+\alpha)} + \frac{2^{n-1} z^{n+\alpha+1}}{(n+1)(n+\alpha+1)}$$

Involving algebraic functions

05.04.21.0005.01

$$\int (1-z^2)^{\frac{1}{2}(-n-3)} T_n(z) dz = \frac{(1-z^2)^{\frac{1}{2}(-n-1)}}{n+1} T_{n+1}(z)$$

05.04.21.0006.01

$$\int (1-z^2)^{\frac{n-3}{2}} T_n(z) dz = -\frac{(1-z^2)^{\frac{n-1}{2}}}{n-1} T_{n-1}(z)$$

Definite integration

Involving the direct function

05.04.21.0007.01

$$\mathcal{P} \int_{-1}^1 \frac{T_n(t)}{\sqrt{1-t^2} (t-x)} dt = \pi U_{n-1}(x) ; n > 1 \wedge -1 < x < 1$$

05.04.21.0008.01

$$\int_{-1}^1 T_n(x)^2 \log(T_n(x)^2) \frac{1}{\sqrt{1-x^2}} dx = -\pi (1 - \delta_{n,0}) (\log(2) - 1) ; n \in \mathbb{N}$$

Entropy integral

Orthogonality:

05.04.21.0009.01

$$\int_{-1}^1 \frac{T_m(t) T_n(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2} \delta_{m,n} ; m^2 + n^2 \neq 0$$

Summation

Infinite summation

05.04.23.0001.01

$$\sum_{n=0}^{\infty} T_n(z) w^n = \frac{1-wz}{w^2-2zw+1} ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0002.01

$$\sum_{n=1}^{\infty} \frac{w^n}{n} T_n(z) = -\frac{1}{2} \log(w^2 - 2zw + 1) ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0003.01

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} T_n(z) w^n = \frac{\sqrt{1-wz + \sqrt{w^2 - 2zw + 1}}}{\sqrt{2} \sqrt{w^2 - 2zw + 1}} ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0004.01

$$\sum_{n=0}^{\infty} \frac{T_n(z) w^n}{\left(\frac{1}{2}\right)_n n!} = \cosh(\sqrt{2} \sqrt{w(z-1)}) \cosh(\sqrt{2} \sqrt{w(z+1)}) ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0005.01

$$\sum_{n=0}^{\infty} \frac{T_n(z) w^n}{n!} = e^{wz} \cosh(\sqrt{w^2(z^2-1)}) ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n (-\gamma)_n}{\left(\frac{1}{2}\right)_n n!} T_n(z) w^n = \cos \left(2\gamma \sin^{-1} \left(\frac{\sqrt{-w - \sqrt{w^2 - 2zw + 1}} + 1}{\sqrt{2}} \right) \right) \cos \left(2\gamma \sin^{-1} \left(\frac{\sqrt{w - \sqrt{w^2 - 2zw + 1}} + 1}{\sqrt{2}} \right) \right) ;$$

$$-1 < z < 1 \wedge |w| < 1$$

05.04.23.0007.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} T_n(z) w^n = (1-wz)^{-\gamma} {}_2F_1 \left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{1}{2}; \frac{(z^2-1)w^2}{(1-wz)^2} \right) ; -1 < z < 1 \wedge |w| < 1$$

05.04.23.0008.01

$$\sum_{n=1}^{\infty} \frac{1}{2^n \prod_{k=1}^{n-1} T_k\left(\frac{z}{2}\right)} = \frac{1}{2} \left(z - \sqrt{z^2 - 4} \right) ; |z| > 2$$

05.04.23.0009.01

$$\sum_{n=0}^{\infty} T_n(x) T_n(y) = \frac{1}{2} \pi \sqrt[4]{1-x^2} \sqrt[4]{1-y^2} \delta(x-y) ; -1 < x < 1 \wedge -1 < y < 1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $T_n(x)$, $n = 0, 1, \dots$, forms a complete, orthogonal (with weight $\frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}$) system on the interval $(-1, 1)$.

05.04.25.0001.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{T_n(x)}{\sqrt{1-x^2}} \right) \left(\sqrt{\frac{2}{\pi}} \frac{T_n(y)}{\sqrt{1-y^2}} \right) = \delta(x-y) ; -1 < x < 1 \wedge -1 < y < 1$$

05.04.25.0002.01

$$\int_{-1}^1 \left(\sqrt{\frac{2}{\pi}} \frac{T_m(t)}{\sqrt{1-t^2}} \right) \left(\sqrt{\frac{2}{\pi}} \frac{T_n(t)}{\sqrt{1-t^2}} \right) dt = \delta_{m,n} ; m^2 + n^2 \neq 0$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{T_n(x)\}_{n=0,1,\dots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.04.25.0003.01

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \psi_n(x) ; c_n = \int_{-1}^1 \psi_n(t) f(t) dt \wedge \psi_n(x) = \sqrt{\frac{2}{\pi}} \frac{T_n(x)}{\sqrt{1-x^2}} \wedge -1 < x < 1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

05.04.26.0001.01

$$T_n(z) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-z}{2}\right)$$

05.04.26.0002.01

$$T_n(z) = (-1)^n {}_2F_1\left(-n, n; \frac{1}{2}; \frac{z+1}{2}\right)$$

Through hypergeometric functions of two variables

05.04.26.0003.01

$$T_n(z) = F_{1 \times 0 \times 0}^{2 \times 0 \times 0}\left(-n, n;;; \frac{1}{2}, -\frac{z}{2}\right)$$

Through Meijer G

Classical cases for the direct function itself

05.04.26.0004.01

$$T_n(z) = -\frac{1}{\sqrt{\pi}} \lim_{m \rightarrow n} m \sin(\pi m) G_{2,2}^{1,2} \left(\frac{z-1}{2} \left| \begin{matrix} m+1, 1-m \\ 0, \frac{1}{2} \end{matrix} \right. \right)$$

Classical cases involving algebraic functions

05.04.26.0005.01

$$(z+1)^{-n} T_n \left(\frac{1-z}{1+z} \right) = \frac{2^{2n-1}}{\Gamma(2n)} G_{2,2}^{1,2} \left(z \left| \begin{matrix} \frac{1}{2}-n, 1-n \\ 0, \frac{1}{2} \end{matrix} \right. \right); z \notin (-\infty, -1)$$

05.04.26.0006.01

$$(z+1)^{-n} T_n \left(\frac{z-1}{z+1} \right) = \frac{2^{2n-1}}{\Gamma(2n)} G_{2,2}^{2,1} \left(z \left| \begin{matrix} 1-n, \frac{1}{2}-n \\ 0, \frac{1}{2} \end{matrix} \right. \right); z \notin (-1, 0)$$

05.04.26.0007.01

$$(z+1)^{-\frac{n}{2}} T_n \left(\frac{1}{\sqrt{z+1}} \right) = \frac{2^{n-1}}{\Gamma(n)} G_{2,2}^{1,2} \left(z \left| \begin{matrix} 1-\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right)$$

05.04.26.0008.01

$$(z+1)^{-\frac{n}{2}} T_n \left(\sqrt{\frac{z}{z+1}} \right) = \frac{2^{n-1}}{\Gamma(n)} G_{2,2}^{2,1} \left(z \left| \begin{matrix} 1-\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right); z \notin (-1, 0)$$

Classical cases involving unit step θ

05.04.26.0009.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}} T_n(2z-1) = \sqrt{\pi} G_{2,2}^{2,0} \left(z \left| \begin{matrix} \frac{1}{2}-n, n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right); z \notin (-1, 0)$$

05.04.26.0010.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}} T_n(2z-1) = \sqrt{\pi} G_{2,2}^{0,2} \left(z \left| \begin{matrix} \frac{1}{2}-n, n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right)$$

05.04.26.0011.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}} T_n \left(\frac{2}{z} - 1 \right) = \sqrt{\pi} G_{2,2}^{2,0} \left(z \left| \begin{matrix} 0, \frac{1}{2} \\ -n, n \end{matrix} \right. \right)$$

05.04.26.0012.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}} T_n \left(\frac{2}{z} - 1 \right) = \sqrt{\pi} G_{2,2}^{0,2} \left(z \left| \begin{matrix} 0, \frac{1}{2} \\ -n, n \end{matrix} \right. \right); z \notin (-\infty, -1)$$

05.04.26.0013.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}} T_n(8z^2-8z+1) = \sqrt{\pi} G_{2,2}^{0,2} \left(z \left| \begin{matrix} \frac{1}{2}-2n, 2n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right); \operatorname{Re}(z) > 0$$

05.04.26.0014.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}} T_n \left(\frac{8}{z^2} - \frac{8}{z} + 1 \right) = \sqrt{\pi} G_{2,2}^{2,0} \left(z \left| \begin{matrix} 0, \frac{1}{2} \\ -2n, 2n \end{matrix} \right. \right); \operatorname{Re}(z) > 0$$

Generalized cases involving algebraic functions

05.04.26.0015.01

$$(z^2 + 1)^{-\frac{n}{2}} T_n \left(\frac{z}{\sqrt{z^2 + 1}} \right) = \frac{2^{n-1}}{\Gamma(n)} G_{2,2}^{2,1} \left(z, \frac{1}{2} \left| \begin{matrix} 1 - \frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix} \right. \right) ; \operatorname{Re}(z) > 0$$

Generalized cases involving unit step θ

05.04.26.0016.01

$$\frac{\theta(|z| - 1)}{\sqrt{z^2 - 1}} T_n(z) = \sqrt{\pi} \sqrt{z^2} G_{2,2}^{0,2} \left(z, \frac{1}{2} \left| \begin{matrix} -\frac{n}{2}, \frac{n}{2} \\ -\frac{1}{2}, 0 \end{matrix} \right. \right)$$

05.04.26.0017.01

$$\frac{\theta(1 - |z|)}{\sqrt{1 - z^2}} T_n \left(\frac{1}{z} \right) = \sqrt{\pi} G_{2,2}^{2,0} \left(z, \frac{1}{2} \left| \begin{matrix} 0, \frac{1}{2} \\ -\frac{n}{2}, \frac{n}{2} \end{matrix} \right. \right)$$

05.04.26.0018.01

$$\frac{\theta(|z| - 1)}{\sqrt{z^2 - 1}} T_n \left(\frac{1}{z} \right) = \sqrt{\pi} \sqrt{z^2} G_{2,2}^{0,2} \left(z, \frac{1}{2} \left| \begin{matrix} -\frac{1}{2}, 0 \\ -\frac{n+1}{2}, \frac{n-1}{2} \end{matrix} \right. \right)$$

Through other functions**Involving some hypergeometric-type functions**

05.04.26.0019.01

$$T_n(z) = \sqrt{\frac{\pi}{2}} \sqrt[4]{1 - z^2} P_{n-\frac{1}{2}}^{\frac{1}{2}}(z)$$

05.04.26.0020.01

$$T_n(z) = \sqrt{\frac{\pi}{2}} \sqrt[4]{z - 1} \sqrt[4]{z + 1} P_{n-\frac{1}{2}}^{\frac{1}{2}}(z)$$

05.04.26.0021.01

$$T_n(z) = \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{\left(-\frac{1}{2}, -\frac{1}{2}\right)}(z)$$

05.04.26.0022.01

$$T_n(z) = \frac{P_n^{\left(-\frac{1}{2}, -\frac{1}{2}\right)}(z)}{P_n^{\left(-\frac{1}{2}, -\frac{1}{2}\right)}(1)}$$

Involving spheroidal functions

05.04.26.0023.01

$$T_n(z) = \frac{\sqrt{2} \sqrt[4]{1 - z^2}}{\sqrt{\pi}} PS_{n-\frac{1}{2}, \frac{1}{2}}(0, z)$$

Representations through equivalent functions**With related functions**

05.04.27.0001.01

$$T_n(z) = \frac{n}{2} C_n^{(0)}(z)$$

05.04.27.0002.01

$$T_n(z) = U_n(z) - z U_{n-1}(z)$$

05.04.27.0003.01

$$T_n(z) = \frac{1}{2} (U_n(z) - U_{n-2}(z))$$

05.04.27.0004.01

$$T_n(z) = -\frac{1}{n} \left(\frac{\partial((z^2 - 1) U_{n-1}(z))}{\partial z} - z U_{n-1}(z) \right)$$

05.04.27.0005.01

$$T_n(x) = -\frac{1}{\pi} \mathcal{P} \int_{-1}^1 \frac{\sqrt{1-t^2} U_{n-1}(t)}{t-x} dt ; n > 0 \wedge -1 < x < 1$$

05.04.27.0010.01

$$T_{2n+1}(z) = (-1)^n z U_{2n}(\sqrt{1-z^2})$$

With elementary functions

05.04.27.0006.01

$$T_n(z) = \frac{1}{2} \left(e^{\frac{i\pi n}{2}} \left(i z + \sqrt{1-z^2} \right)^{-n} + e^{-\frac{i\pi n}{2}} \left(i z + \sqrt{1-z^2} \right)^n \right)$$

05.04.27.0007.01

$$T_n(z) = \cos(n \cos^{-1}(z))$$

05.04.27.0011.01

$$T_n(z) = (-1)^n \cos \left(2n \sin^{-1} \left(\frac{\sqrt{z+1}}{\sqrt{2}} \right) \right)$$

05.04.27.0008.01

$$T_n(z) = 2^{-n-1} (-1)^n \left((\sqrt{1-z} - \sqrt{-z-1})^{2n} + (\sqrt{1-z} + \sqrt{-z-1})^{2n} \right)$$

05.04.27.0009.01

$$T_n(z) = \frac{1}{2} z^n \left(\left(1 - \sqrt{1 - \frac{1}{z^2}} \right)^n + \left(1 + \sqrt{1 - \frac{1}{z^2}} \right)^n \right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k \psi_k(x) ; c_k = \int_{-1}^1 f(t) \psi_k(t) dt, \psi_k(x) = \sqrt{\frac{2}{\pi}} (1-x^2)^{-\frac{1}{4}} T_k(x), k \in \mathbb{N}.$$

Minimizing property of Chebyshev polynomials

Chebyshev polynomials $T_n(x)$ have the smallest absolute values among all polynomials of degree n with leading coefficient 1:

$$\max_{-1 \leq x \leq 1} \left| \sum_{k=0}^n a_k x^k \right| \geq \max_{-1 \leq x \leq 1} \left| \frac{T_n(x)}{[x^n] (T_n(x))} \right|.$$

Another minimizing property of Chebyshev polynomials

$$\int_{-1}^1 \frac{P(x)^2}{\sqrt{1-x^2}} dx \geq \int_{-1}^1 \frac{T_n(x)^2}{\sqrt{1-x^2}} dx = \frac{\pi}{2} /; \quad P(x) = 2^{n-1} x^n + \sum_{k=0}^{n-1} c_k x^k \bigwedge c_k \in \mathbb{R}$$

Distribution of the zeros of Chebyshev polynomials of high order

In the limit $n \rightarrow \infty$, the zeros of $T_n(x)$ fulfill the arcsin distribution; in other words, the relative number of zeros $m_n(a, b)/n$ in the interval (a, b) is $(\arcsin(b) - \arcsin(a))/\pi$.

History

– P. L. Chebyshev (1854,1855,1859)

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