

WeierstrassInvariants

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Notations

Traditional name

Weierstrass invariants

Traditional notation

 $\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$

Mathematica StandardForm notation

WeierstrassInvariants[$\{\omega_1, \omega_3\}$]

Primary definition

09.19.02.0001.02

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ 60 \sum_{\substack{m, n = -\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, 140 \sum_{\substack{m, n = -\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^6} \right\} /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

Specific values

Values at infinities

09.19.03.0001.01

$$g_2(\omega_1, \infty)^3 - 27 g_3(\omega_1, \infty)^2 = 0$$

09.19.03.0002.01

$$\frac{9 g_3(\omega_1, \infty)}{2 g_2(\omega_1, \infty)} = \left(\frac{\pi}{2 \omega_1}\right)^2$$

09.19.03.0003.01

$$\{g_2(\infty, \infty), g_3(\infty, \infty)\} = \{0, 0\}$$

General characteristics

Domain and analyticity

$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ is an vector-valued function of ω_1 and ω_3 that is analytic in each component and it is defined over $\mathbb{C}^2$ (for $\omega_1 \neq a \omega_3, a \in \mathbb{R}$).

09.19.04.0001.01

$$\{\omega_1 * \omega_3\} \longrightarrow \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \longrightarrow \{\mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.19.04.0002.01

$$\{g_2(\overline{\omega_1}, \overline{\omega_3}), g_3(\overline{\omega_1}, \overline{\omega_3})\} = \overline{\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}$$

Permutation symmetry

09.19.04.0003.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_3, \omega_1), g_3(\omega_3, \omega_1)\}$$

Periodicity

$\{g_2(1, \omega_3), g_3(1, \omega_3)\}$ is a periodic vector-valued function with period 1 .

09.19.04.0007.01

$$\{g_2(1, \omega_3 + 1), g_3(1, \omega_3 + 1)\} = \{g_2(1, \omega_3), g_3(1, \omega_3)\}$$

09.19.04.0004.01

$$\{g_2(1, \omega_3 + n), g_3(1, \omega_3 + n)\} = \{g_2(1, \omega_3), g_3(1, \omega_3)\} ; n \in \mathbb{Z}$$

09.19.04.0008.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_1, \omega_3 + n \omega_1), g_3(\omega_1, \omega_3 + n \omega_1)\} ; n \in \mathbb{Z}$$

09.19.04.0009.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_1 + n \omega_3, \omega_3), g_3(\omega_1 + n \omega_3, \omega_3)\} ; n \in \mathbb{Z}$$

Transformation of half-periods

09.19.04.0005.01

$$\{g_2(\alpha \omega_1 + \beta \omega_3, \gamma \omega_1 + \delta \omega_3), g_3(\alpha \omega_1 + \beta \omega_3, \gamma \omega_1 + \delta \omega_3)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} ; \{\alpha, \beta, \gamma, \delta\} \in \mathbb{Z} \wedge \alpha \delta - \beta \gamma = \pm 1$$

Homogeneity

09.19.04.0006.01

$$\{g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)\} = \left\{ \frac{1}{\lambda^4} g_2(\omega_1, \omega_3), \frac{1}{\lambda^6} g_3(\omega_1, \omega_3) \right\} ; \lambda \in \mathbb{C}$$

09.19.04.0010.01

$$\left\{ g_2\left(1, \frac{\omega_3}{\omega_1}\right), g_3\left(1, \frac{\omega_3}{\omega_1}\right) \right\} = \{\omega_1^4, \omega_1^6\} \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$$

Series representations

Generalized power series

09.19.06.0001.02

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ 60 \sum_{\substack{m, n = -\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, 140 \sum_{\substack{m, n = -\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^6} \right\} ; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

q-series

09.19.06.0002.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ 20 \left(\frac{\pi}{2\omega_1} \right)^4 \left(\frac{1}{15} + 16 \sum_{k=1}^{\infty} \frac{k^3 q^{2k}}{1 - q^{2k}} \right), 28 \left(\frac{\pi}{2\omega_1} \right)^6 \left(\frac{2}{189} - \frac{16}{3} \sum_{k=1}^{\infty} \frac{k^5 q^{2k}}{1 - q^{2k}} \right) \right\} ; q = \exp\left(\pi i \frac{\omega_3}{\omega_1}\right)$$

Other series representations

09.19.06.0003.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{\pi^4}{12} \left(\frac{1}{\omega_3^4} + \frac{1}{\omega_1^4} \right) + \frac{15}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{(m\omega_1 - n\omega_3)^4} + \frac{1}{(m\omega_1 + n\omega_3)^4} \right), \right. \\ \left. \frac{\pi^6}{216} \left(\frac{1}{\omega_3^6} + \frac{1}{\omega_1^6} \right) + \frac{35}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{(m\omega_1 - n\omega_3)^6} + \frac{1}{(m\omega_1 + n\omega_3)^6} \right) \right\} ; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

09.19.06.0004.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{\pi^4}{12} \frac{1}{\omega_1^4} + \frac{15}{2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_3)^4}, \frac{\pi^6}{216} \frac{1}{\omega_1^6} + \frac{35}{8} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_3)^6} \right\} ; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

09.19.06.0005.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{\pi^4}{12} \frac{1}{\omega_3^4} + \frac{15}{2} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_3)^4}, \frac{\pi^6}{216} \frac{1}{\omega_3^6} + \frac{35}{8} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_3)^6} \right\} ; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

09.19.06.0006.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{\pi^4}{\omega_1^4} \left(20 \sum_{k=1}^{\infty} \sigma_3(k) q^{2k} + \frac{1}{12} \right), \frac{\pi^6}{\omega_1^6} \left(\frac{1}{216} - \frac{7}{3} \sum_{k=1}^{\infty} \sigma_5(k) q^{2k} \right) \right\} ; q = \exp\left(\frac{i\pi\omega_2}{\omega_1}\right) \wedge \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

Integral representations

On the real axis

Of the direct function

09.19.07.0001.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{5}{8} \int_0^{\infty} t^3 (u(t, \omega_1, \omega_3) + v(t, \omega_1, \omega_3)) dt, \frac{7}{384} \int_0^{\infty} t^5 (u(t, \omega_1, \omega_3) - v(t, \omega_1, \omega_3)) dt \right\} ; \\ u(t, \omega_1, \omega_3) = \frac{\cosh(t\omega_3) + e^{-\frac{t}{2}\omega_3} \sinh\left(\frac{t\omega_3}{2}\right)}{\sinh\left(\frac{1}{2}t(\omega_1 - \omega_3)\right) \sinh\left(\frac{1}{2}t(\omega_1 + \omega_3)\right)} \wedge v(t, \omega_1, \omega_3) = \frac{e^{\frac{it}{2}\omega_3} \cos\left(\frac{t\omega_3}{2}\right)}{\sin\left(\frac{1}{2}t(\omega_1 - \omega_3)\right) \sin\left(\frac{1}{2}t(\omega_1 + \omega_3)\right)}$$

Differential equations

Partial differential equations

09.19.13.0001.01

$$\omega_1 \frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} + \omega_3 \frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3} + \{4, 6\} \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = 0$$

09.19.13.0002.01

$$\omega_1 \frac{\partial g_2(\omega_1, \omega_3)}{\partial \omega_1} + \omega_3 \frac{\partial g_2(\omega_1, \omega_3)}{\partial \omega_3} + 4 g_2(\omega_1, \omega_3) = 0$$

09.19.13.0003.01

$$\omega_1 \frac{\partial g_3(\omega_1, \omega_3)}{\partial \omega_1} + \omega_3 \frac{\partial g_3(\omega_1, \omega_3)}{\partial \omega_3} + 6 g_3(\omega_1, \omega_3) = 0$$

Transformations

Transformations and argument simplifications

09.19.16.0001.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{\omega_1^{-4}, \omega_1^{-6}\} \left\{ g_2\left(1, \frac{\omega_3}{\omega_1}\right), g_3\left(1, \frac{\omega_3}{\omega_1}\right) \right\}$$

09.19.16.0002.01

$$\{g_2(\omega_1, \omega_3 + n \omega_1), g_3(\omega_1, \omega_3 + n \omega_1)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \text{ ; } n \in \mathbb{Z}$$

09.19.16.0003.01

$$\{g_2(\omega_1 + n \omega_3, \omega_3), g_3(\omega_1 + n \omega_3, \omega_3)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \text{ ; } n \in \mathbb{Z}$$

Identities

Functional identities

09.19.17.0001.01

$$\Delta(\tau)^6 = \frac{1}{16777216} \left(\Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^4 + 2 \Delta\left(\frac{\tau}{2}\right)^3 \Delta\left(\frac{\tau+1}{2}\right)^3 + \Delta\left(\frac{\tau}{2}\right)^4 \Delta\left(\frac{\tau+1}{2}\right)^2 - 2304 \Delta(\tau)^2 \Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^2 + 393216 \Delta\left(\frac{\tau}{2}\right) \Delta(\tau)^4 \Delta\left(\frac{\tau+1}{2}\right) \right) \text{ ; } \Delta(\tau) = g_2^3 - 27 g_3^2 \wedge \{g_2, g_3\} = \{g_2(1, \tau), g_3(1, \tau)\}$$

Differentiation

Low-order differentiation

With respect to ω_1

09.19.20.0015.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\left\{12, \frac{2}{3}\right\} \text{Reverse}\left[\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}^{[2,1]}\right] \omega_3 - \{8, 12\} \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \zeta(\omega_3; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) \right)$$

09.19.20.0001.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} = \left\{ -30 \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m \omega_1 + n \omega_3)^5}, -\frac{105}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m \omega_1 + n \omega_3)^7} \right\}$$

09.19.20.0002.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} = \left\{ -\frac{4g_2}{\omega_1} - \frac{40i\pi^5\omega_3}{\omega_1^6} \sum_{k=1}^{\infty} \frac{k^4 q^{2k}}{(1-q^{2k})^2}, \frac{14i\pi^7\omega_3}{3\omega_1^8} \sum_{k=1}^{\infty} \frac{k^6 q^{2k}}{(1-q^{2k})^2} - \frac{6g_3}{\omega_1} \right\}; q = \exp\left(\pi i \frac{\omega_3}{\omega_1}\right)$$

09.19.20.0003.01

$$\frac{\partial^2 \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1^2} = \left\{ 150 \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m^2}{(m\omega_1 + n\omega_3)^6}, \frac{735}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m^2}{(m\omega_1 + n\omega_3)^8} \right\}$$

09.19.20.0011.02

$$\frac{\partial g_2}{\partial \omega_1} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} (12g_3\omega_3 - 8g_2\eta_3); \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \wedge \eta_3 = \zeta(\omega_3; g_2, g_3)$$

09.19.20.0012.02

$$\frac{\partial g_3}{\partial \omega_1} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\frac{2}{3} g_2^2 \omega_3 - 12g_3\eta_3 \right); \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \wedge \eta_3 = \zeta(\omega_3; g_2, g_3)$$

With respect to ω_3

09.19.20.0016.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\left\{ 12, \frac{2}{3} \right\} \text{Reverse}[\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}^{[2,1]}] \omega_1 - \{8, 12\} \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \zeta(\omega_1; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) \right)$$

09.19.20.0004.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3} = \left\{ -30 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n}{(m\omega_1 + n\omega_3)^5}, -\frac{105}{4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n}{(m\omega_1 + n\omega_3)^7} \right\}$$

09.19.20.0005.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3} = \left\{ \frac{40i\pi^5}{\omega_1^5} \sum_{k=1}^{\infty} \frac{q^{2k} k^4}{(1-q^{2k})^2}, -\frac{14i\pi^7}{3\omega_1^7} \sum_{k=1}^{\infty} \frac{q^{2k} k^6}{(1-q^{2k})^2} \right\}; q = \exp\left(\pi i \frac{\omega_3}{\omega_1}\right)$$

09.19.20.0006.01

$$\frac{\partial^2 \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3^2} = \left\{ 150 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n^2}{(m\omega_1 + n\omega_3)^6}, \frac{735}{4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n^2}{(m\omega_1 + n\omega_3)^8} \right\}$$

09.19.20.0013.02

$$\frac{\partial g_2}{\partial \omega_3} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} (12g_3\omega_1 - 8g_2\eta_1); \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \wedge \eta_1 = \zeta(\omega_1; g_2, g_3)$$

09.19.20.0014.02

$$\frac{\partial g_3}{\partial \omega_3} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\frac{2}{3} g_2^2 \omega_1 - 12g_3\eta_1 \right); \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \wedge \eta_1 = \zeta(\omega_1; g_2, g_3)$$

Symbolic differentiation**With respect to ω_1**

09.19.20.0007.01

$$\frac{\partial^k \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1^k} = \left\{ \frac{5}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k (k+3)! m^k}{(m \omega_1 + n \omega_3)^{4+k}}, \frac{7}{192} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k (k+5)! m^k}{(m \omega_1 + n \omega_3)^{6+k}} \right\} /; k \in \mathbb{N}^+$$

With respect to ω_3

09.19.20.0008.01

$$\frac{\partial^k \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_3^k} = \left\{ \frac{5}{4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^k (k+3)! n^k}{(m \omega_1 + n \omega_3)^{4+k}}, \frac{7}{192} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^k (k+5)! n^k}{(m \omega_1 + n \omega_3)^{6+k}} \right\} /; k \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to ω_1

09.19.20.0009.01

$$\left\{ \frac{\partial^\alpha g_2(\omega_1, \omega_3)}{\partial \omega_1^\alpha}, \frac{\partial^\alpha g_3(\omega_1, \omega_3)}{\partial \omega_1^\alpha} \right\} = \left\{ \frac{\pi^4}{12} \mathcal{FC}_{\exp}^{(\alpha)}(\omega_1, -4) \omega_1^{-4-\alpha} + \frac{15 \omega_1^{-\alpha}}{2 \omega_3^4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^4} {}_2\tilde{F}_1\left(1, 4; 1-\alpha; -\frac{m \omega_1}{n \omega_3}\right), \right. \\ \left. \frac{\pi^6}{216} \mathcal{FC}_{\exp}^{(\alpha)}(\omega_1, -6) \omega_1^{-6-\alpha} + \frac{35 \omega_1^{-\alpha}}{8 \omega_3^6} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^6} {}_2\tilde{F}_1\left(1, 6; 1-\alpha; -\frac{m \omega_1}{n \omega_3}\right) \right\}$$

With respect to ω_3

09.19.20.0010.01

$$\left\{ \frac{\partial^\alpha g_2(\omega_1, \omega_3)}{\partial \omega_3^\alpha}, \frac{\partial^\alpha g_3(\omega_1, \omega_3)}{\partial \omega_3^\alpha} \right\} = \left\{ \frac{\pi^4}{12} \mathcal{FC}_{\exp}^{(\alpha)}(\omega_3, -4) \omega_3^{-4-\alpha} + \frac{15 \omega_3^{-\alpha}}{2 \omega_1^4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^4} {}_2\tilde{F}_1\left(1, 4; 1-\alpha; -\frac{n \omega_3}{m \omega_1}\right), \right. \\ \left. \frac{\pi^6}{216} \mathcal{FC}_{\exp}^{(\alpha)}(\omega_3, -6) \omega_3^{-6-\alpha} + \frac{35 \omega_3^{-\alpha}}{8 \omega_1^6} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^6} {}_2\tilde{F}_1\left(1, 6; 1-\alpha; -\frac{n \omega_3}{m \omega_1}\right) \right\}$$

Integration

Indefinite integration

Involving only one direct function with respect to ω_1

09.19.21.0001.01

$$\left\{ \int g_2(\omega_1, \omega_3) d\omega_1, \int g_3(\omega_1, \omega_3) d\omega_1 \right\} = \left\{ \frac{5 \omega_1}{2 \omega_3^3} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 \omega_1^2 + 3 m n \omega_3 \omega_1 + 3 n^2 \omega_3^2}{n^3 (m \omega_1 + n \omega_3)^3} - \frac{\pi^4}{36 \omega_1^3}, \right. \\ \left. \frac{7 \omega_1}{8 \omega_3^5} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (m^4 \omega_1^4 + 5 m^3 n \omega_3 \omega_1^3 + 10 m^2 n^2 \omega_3^2 \omega_1^2 + 10 m n^3 \omega_3^3 \omega_1 + 5 n^4 \omega_3^4) / (n^5 (m \omega_1 + n \omega_3)^5) - \frac{\pi^6}{1080 \omega_1^5} \right\}$$

Representations through equivalent functions

With inverse function

09.19.27.0001.01

$$\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

With related functions

Involving Weierstrass functions

09.19.27.0002.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{-4(e_1 e_2 + e_3 e_2 + e_1 e_3), 4 e_1 e_2 e_3\} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.19.27.0003.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{2(e_1^2 + e_2^2 + e_3^2), 4 e_1 e_2 e_3\} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.19.27.0004.01

$$\{g_2, g_3\} = \{-4(e_1 e_3 + e_2(e_1 + e_3)), 4 e_1 e_3 e_2\} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.19.27.0005.01

$$\frac{g_2^2}{8} = e_1^4 + e_2^4 + e_3^4 /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.19.27.0006.01

$$4 e_k^3 - g_2 e_k - g_3 = 0 /;$$

$$k \in \{1, 2, 3\} \wedge \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.19.27.0007.01

$$g_3^3 - 27 g_2^2 = 16(e_1 - e_2)^2(e_3 - e_1)^2(e_3 - e_2)^2 /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

Involving theta functions

09.19.27.0008.01

$$g_2(\omega_1, \omega_3) = \frac{1}{12} \left(\frac{\pi}{\omega_1} \right)^4 (\vartheta_2(0, q)^8 - \vartheta_3(0, q)^4 \vartheta_2(0, q)^4 + \vartheta_3(0, q)^8) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0009.01

$$g_2(\omega_1, \omega_3) = \frac{2}{3} \left(\frac{\pi}{2 \omega_1} \right)^4 (\vartheta_2(0, q)^8 + \vartheta_3(0, q)^8 + \vartheta_4(0, q)^8) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0010.01

$$g_3(\omega_1, \omega_3) = \left(\frac{\pi}{2 \omega_1} \right)^6 \left(\frac{8}{27} (\vartheta_2(0, q)^{12} + \vartheta_3(0, q)^{12}) - \frac{4}{9} (\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4) \vartheta_2(0, q)^4 \vartheta_3(0, q)^4 \right) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0011.01

$$g_3(\omega_1, \omega_3) = \frac{4}{27} \left(\frac{\pi}{2 \omega_1} \right)^6 (\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4) (\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4) (\vartheta_4(0, q)^4 - \vartheta_2(0, q)^4) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

Involving elliptic integrals and modular functions

09.19.27.0012.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{ \frac{4(m^2 - m + 1)K(m)^4}{3\omega_1^4}, \frac{4(m-2)(2m-1)(m+1)K(m)^6}{27\omega_1^6} \right\} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

09.19.27.0013.01

$$g_2^3 (2m^3 - 3m^2 - 3m + 2)^2 - 108 g_3^2 (m^2 - m + 1)^3 \neq 0; m = q^{-1} \left(\exp \left(\frac{i\pi \omega_3}{\omega_1} \right) \right) \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Theorems

Transcendentality of one expression with Weierstrass functions

Any $\alpha_1 \omega_1(g_2, g_3) + \alpha_2 \omega_3(g_2, g_3) + \alpha_3 \zeta(\omega_1(g_2, g_3); g_2, g_3) + \alpha_4 \zeta(\omega_3(g_2, g_3); g_2, g_3) + \alpha_5 2\pi i \neq 0$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are algebraic numbers, is transcendental.

History

- A. Cayley
- G. Boole (1845)

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